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First Regularized Trace of the Limit Assignment of Sturm-Liouville Type with Two Constant Delays

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Abstract. We observe spectral assignment $D^2y = \lambda y$ defined by

$$-y''(x) + q_1(x)y(x-\tau_1) + q_2(x)y(x-\tau_2) = \lambda y(x),\ \lambda = z^2 \tag{1}$$

$$q_1(x),\ q_2(x) \in L_1[0,\pi],\ \tau_1, \tau_2 \in (0,\pi)$$

$$y(x-\tau_1) \equiv 0, x \in (0,\tau_1],\ \tau_1 = k_0\tau_2 \tag{2}$$

$$y(\pi) = 0 \tag{3}$$

In this paper, we construct a solution $y(x,z)$ which satisfies (1) and (2), and then (3) is used to construct the characteristic function $F(z)$, $z \in C$. Then the asymptotics of eigenvalues of the operator D^2 is constructed. Finally, the first regularized trace is calculated.

1. Introduction

As it is known, the trace of a finite-dimensional matrix is the sum of all the eigenvalues. But in an infinitedimensional space, in general, ordinary differential operators do not have a finite trace. Gelfand and Levitan [5] firstly obtained a trace formula for a self-adjoint Sturm-Liouville differential equation. For the scalar Sturm-Liouville problems, there is an enormous literature on estimates of large eigenvalues and regularized trace formula which may often be computed explicitly in terms of the coefficients of operators and boundary conditions. A detailed list of publications related to the present aspect can be found in [3]. A trace formula for the limit assignment of Sturm-Liouville type with two constant delays has never been considered before.

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2. Construction solutions

Theorem 2.1. *If the* $q_i \in L_1[0,\pi]$, $i = 1,2$ *and* $0 < \tau_2 < 2\tau_2 < \ldots < k_0\tau_2 = \tau_1 < \ldots < (l_0k_0 - 1)\tau_2 < l_0\tau_1 = l_0k_0\tau_2 < \pi \le (l_0k_0 + 1)\tau_2$, $(k_0, l_0 \in N)$, *then the solution of the equation (1) with the initial condition (2) has the following form*

$$y(x,z) = \sin zx + \sum_{k=1}^{l_0k_0} \frac{1}{z^k} b^{(2)}_{s^{k+1}}(x,z) + \sum_{k=1}^{l_0} \frac{1}{z^k} b^{(1)}_{s^{k+1}}(x,z) + \sum_{i=1}^{l_0-1} \sum_{j=1}^{(l_0-i)k_0} \frac{1}{z^{j+i}} \sum_{P\in S_{j+i}(i,j)} b^{P}_{s^{j+i+1}}(x,z) \tag{4}$$

wherin

$$b^{(i)}_{s^{k+1}}(x,z) = \int\limits_{k\tau_i}^{x} q_i(t_1)\sin z(x-t_1) b^{(i)}_{s^k}(t_1-\tau_i,z)dt_1,\ b^{(i)}_{s^2}(x,z) = \int\limits_{\tau_i}^{x} q_i(t_1)\sin z(x-t_1)\sin z(t_1-\tau_i)dt_1$$

$i = 1,2;\ k = 2,3,...l_0k_o$, *P permutations with repeating elements 1 and 2.*

Proof. Solving the equation (1) with limit conditions $y(0) = 0$ by using the method of variation of constants, we obtain the integral equation

$$y(x,z) = \sin zx + \frac{1}{z}\int\limits_0^x q_1(t_1)\sin z(x-t_1)y(t_1-\tau_1)dt_1 + \frac{1}{z}\int\limits_0^x q_2(t_1)\sin z(x-t_1)y(t_1-\tau_2)dt_1 \tag{5}$$

We introduce notation

$$y_0(x,z) = \begin{cases} \sin zx, x \ge 0 \\ 0, x < 0. \end{cases}$$

$$y_k(x,z) = \begin{cases} \frac{1}{z}\int\limits_{k\tau_2}^{x} q_2(t_1)\sin z(x-t_1)y_{k-1}(t_1-\tau_2)dt_1 + \frac{1}{z}\int\limits_{k\tau_2}^{x} q_1(t_1)\sin z(x-t_1)y_{k-k_0}(t_1-\tau_1)dt_1,\ x \ge k\tau_2 \\ 0 \qquad ,x < k\tau_2. \end{cases} \tag{6}$$

$k = 1,2,...,l_0k_0$, wherein $y_{k-k_0}(x,z) = 0$ for $k < k_0$, then the solution of the equation (1) with the initial condition $y(x-\tau_1) \equiv 0, x \in (0,\tau_1]$ has the form

$$y(x,z) = \sin zx + \sum_{k=1}^{l_0k_0} y_k(x,z)$$

1. At $[0,\tau_2)$ the solution of the equation (5) is equal to $y_0(x,z) = \sin zx$ because of (6) $y_1(x,z) = 0,\ x < \tau_2$. From (6) we obtain

$$y_1(x,z) = \begin{cases} \frac{1}{z}\int\limits_{\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\sin z(t_1-\tau_2)dt_1 = \frac{1}{z}b^{(2)}_{s^2}(x,z),\ x \ge \tau_2 \\ 0 \qquad ,\ x < \tau_2. \end{cases}$$

$$y_2(x,z) = \begin{cases} \frac{1}{z}\int\limits_{2\tau_2}^{x} q_2(t_1)\sin z(x-t_1)(\frac{1}{z}b^{(2)}_{s^2}(t_1-\tau_2,z))dt_1 = \frac{1}{z^2}b^{(2)}_{s^3}(x,z),\ x \ge 2\tau_2 \\ 0 \qquad ,\ x < 2\tau_2 \end{cases}$$

Induction to prove that

$$y_k(x,z) = \begin{cases} \frac{1}{z^k}b^{(2)}_{s^{k+1}}(x,z),\ x \ge k\tau_2 \\ 0 \quad ,\ x < k\tau_2 \end{cases}$$

for $k \le k_0 - 1$, and is

$$y_{k_0-1}(x,z) = \begin{cases} \frac{1}{z^{k_0-1}} b^{(2)}_{s^{k_0}}(x,z), & x \ge (k_0-1)\tau_2 \\ 0 & , x < (k_0-1)\tau_2. \end{cases}$$

Here we find that the solution is in $[0, k_0\tau_2)$ given by

$$y(x,z) = \sin zx + \sum_{k=1}^{k_0-1} y_k(x,z) = \sin zx + \sum_{k=1}^{k_0-1} \frac{1}{z^k} b^{(2)}_{s^{k+1}}(x,z)$$

2. Since the $[k_0\tau_2, (k_0+1)\tau_2)$ applies $k_0\tau_2 = \tau_1$, in (6) we obtain

$$y_{k_0}(x,z) = \frac{1}{z}\int_{k_0\tau_2}^{x} q_2(t_1)\sin z(x-t_1) y_{k_0-1}(t_1-\tau_2,z)dt_1 + \frac{1}{z}\int_{\tau_1}^{x} q_1(t_1)\sin z(x-t_1) y_0(t_1-\tau_1,z)dt_1$$

$$= \frac{1}{z}\int_{k_0\tau_2}^{x} q_2(t_1)\sin z(x-t_1)\left(\frac{1}{z^{k_0-1}} b^{(2)}_{s^{k_0}}(t_1-\tau_2,z)\right)dt_1 + \frac{1}{z}\int_{\tau_1}^{x} q_1(t_1)\sin z(x-t_1)\sin z(t_1-\tau_1)dt_1$$

i.e.

$$y_{k_0}(x,z) = \begin{cases} \frac{1}{z^{k_0}} b^{(2)}_{s^{k_0+1}}(x,z) + \frac{1}{z} b^{(1)}_{s^2}(x,z), & x \ge k_0\tau_2 \\ 0 & , x < k_0\tau_2. \end{cases}$$

$$y_{k_0+1}(x,z) = \frac{1}{z}\int_{(k_0+1)\tau_2}^{x} q_2(t_1)\sin z(x-t_1) y_{k_0}(t_1-\tau_2,z)dt_1 + \frac{1}{z}\int_{\tau_1+\tau_2}^{x} q_1(t_1)\sin z(x-t_1) y_1(t_1-\tau_1,z)dt_1$$

$$= \frac{1}{z^{k_0+1}} b^{(2)}_{s^{k_0+2}}(x,z) + \frac{1}{z^2} b^{(2,1)}_{s^3}(x,z) + \frac{1}{z^2} b^{(1,2)}_{s^3}(x,z) = \frac{1}{z^{k_0+1}} b^{(2)}_{s^{k_0+2}}(x,z) + \frac{1}{z^2}\sum_{P\in S_2(1,1)} b^{P}_{s^3}(x,z)$$

i.e.

$$y_{k_0+1}(x,z) = \begin{cases} \frac{1}{z^{k_0+1}} b^{(2)}_{s^{k_0+2}}(x,z) + \frac{1}{z^2}\sum\limits_{P\in S_2(1,1)} b^{P}_{s^3}(x,z), & x \ge (k_0+1)\tau_2 \\ 0 & , x < (k_0+1)\tau_2. \end{cases}$$

Induction to prove that is:

$$y_{2k_0-1}(x,z) = \begin{cases} \frac{1}{z^{2k_0-1}} b^{(2)}_{s^{2k_0}}(x,z) + \frac{1}{z^{k_0}}\sum\limits_{P\in S_{k_0}(1,k_0-1)} b^{P}_{s^{k_0+1}}(x,z), & x \ge (2k_0-1)\tau_2 \\ 0 & , x < (2k_0-1)\tau_2. \end{cases}$$

3. As the $2k_0\tau_2 = 2\tau_1$ by using (6) we obtain

$$y_{2k_0}(x,z) = \begin{cases} \frac{1}{z^{2k_0}} b^{(2)}_{s^{2k_0+1}}(x,z) + \frac{1}{z^{k_0+1}}\sum\limits_{P\in S_{k_0+1}(1,k_0)} b^{P}_{s^{k_0+2}}(x,z) + \frac{1}{z^2} b^{(1)}_{s^3}(x,z), & x \ge 2k_0\tau_2 \\ 0 & , x < 2k_0\tau_2. \end{cases}$$

$$y_{2k_0+1}(x,z) = \begin{cases} \frac{1}{z^{2k_0+1}} b^{(2)}_{s^{2k_0+2}}(x,z) + \frac{1}{z^{k_0+2}}\sum\limits_{P\in S_{k_0+2}(1,k_0+1)} b^{P}_{s^{k_0+3}}(x,z) + \frac{1}{z^3}\sum\limits_{P\in S_3(2,1)} b^{P}_{s^4}(x,z), & x \ge (2k_0+1)\tau_2 \\ 0 & , x < (2k_0+1)\tau_2. \end{cases}$$

4. The same way we do

$$y_{3k_0-1}(x,z) = \begin{cases} \frac{1}{z^{3k_0-1}} b^{(2)}_{s^{3k_0}}(x,z) + \frac{1}{z^{2k_0}} \sum_{P\in S_{2k_0}(1,2k_0-1)} b^P_{s^{2k_0+1}}(x,z) + \frac{1}{z^{k_0+1}} \sum_{P\in S_{k_0+1}(2,k_0-1)} b^P_{s^{k_0+2}}(x,z), & x \geq (3k_0-1)\tau_2 \\ 0 & , x < (3k_0-1)\tau_2. \end{cases}$$

$$y_{3k_0}(x,z) = \begin{cases} \frac{1}{z^{3k_0}} b^{(2)}_{s^{3k_0+1}}(x,z) + \frac{1}{z^{2k_0+1}} \sum_{P\in S_{2k_0+1}(1,2k_0)} b^P_{s^{2k_0+2}}(x,z) + \frac{1}{z^{k_0+2}} \sum_{P\in S_{k_0+2}(2,k_0)} b^P_{s^{k_0+3}}(x,z) + \frac{1}{z^2} b^{(1)}_{s^3}(x,z), & x \geq 3k_0\tau_2 \\ 0 & , x < 3k_0\tau_2. \end{cases}$$

5. Based on the foregoing, we conclude that induction on n to prove that valid of formula (7) - (9).
For $m = 2, 3, ...l_0 - 1$ and $n = 2, ...m - 1$

$$y_{mk_0+n}(x,z) = \frac{1}{z^{mk_0+n}} b^{(2)}_{s^{mk_0+n+1}}(x,z) + \frac{1}{z^{(m-1)k_0+n+1}} \sum_{P\in S_{(m-1)k_0+n+1}(1,(m-1)k_0+n)} b^P_{s^{(m-1)k_0+n+2}}(x,z) +$$

$$+ \frac{1}{z^{(m-2)k_0+n+2}} \sum_{P\in S_{(m-2)k_0+n+2}(2,(m-2)k_0+n)} b^P_{s^{(m-2)k_0+n+3}}(x,z) + ... + \frac{1}{z^{k_0+n+m-1}} \sum_{P\in S_{k_0+n+m-1}(m-1,k_0+n)} b^P_{s^{k_0+n+m}}(x,z) + \quad (7)$$

$$+ \frac{1}{z^{n+m}} \sum_{P\in S_{n+m}(m,n)} b^P_{s^{n+m+1}}(x,z),\ x \geq (mk_0+n)\tau_2; \qquad y_{mk_0+n}(x,z) = 0,\ x < (mk_0+n)\tau_2$$

For $n = 1, 2, ..., l_0$

$$y_{nk_0}(x,z) = \frac{1}{z^{nk_0}} b^{(2)}_{s^{nk_0+1}}(x,z) + \frac{1}{z^{(n-1)k_0+1}} \sum_{P\in S_{(n-1)k_0+1}(1,(n-1)k_0)} b^P_{s^{(n-1)k_0+2}}(x,z) +$$

$$+ \frac{1}{z^{(n-2)k_0+2}} \sum_{P\in S_{(n-2)k_0+2}(2,(n-2)k_0)} b^P_{s^{(n-2)k_0+3}}(x,z) +$$

$$... + \frac{1}{z^{k_0+n-1}} \sum_{P\in S_{k_0+n-1}(n-1,k_0)} b^P_{s^{k_0+n}}(x,z) + \frac{1}{z^n} b^{(1)}_{s^{n+1}}(x,z),\ x \geq nk_0\tau_2;$$

$$y_{nk_0}(x,z) = 0\ x < nk_0\tau_2 \quad (8)$$

For $n = 2, 3, ..., l_0 - 1$

$$y_{nk_0+1}(x,z) = \frac{1}{z^{nk_0+1}} b^{(2)}_{s^{nk_0+2}}(x,z) + \frac{1}{z^{(n-1)k_0+2}} \sum_{P\in S_{(n-1)k_0+2}(1,(n-1)k_0+1)} b^P_{s^{(n-1)k_0+3}}(x,z) +$$

$$+ \frac{1}{z^{(n-2)k_0+3}} \sum_{P\in S_{(n-2)k_0+3}(2,(n-2)k_0+1)} b^P_{s^{(n-2)k_0+4}}(x,z) +$$

$$... + \frac{1}{z^{k_0+n}} \sum_{P\in S_{k_0+n}(n-1,k_0+1)} b^P_{s^{k_0+n+1}}(x,z),\ x \geq (nk_0+1)\tau_2;$$

$$y_{nk_0+1}(x,z) = 0\ x < (nk_0+1)\tau_2 \quad (9)$$

6. Let us prove the formula (7). Some of the natural numbers are less than or equal to m and $3 \leq n < m - 2$ true (7). We will show that (7) holds for $n + 1$. From (7) and the recurrence of the formula (6) we obtain

$$y_{mk_0+n+1}(x,z) = \frac{1}{z^{mk_0+n+1}} b^{(2)}_{s^{mk_0+n}}(x,z) +$$

$$+\frac{1}{z^{(m-1)k_0+n+2}}\sum_{P\in S_{(m-1)k_0+n+2}(1,(m-1)k_0+n+1)\setminus(12...2)} b^P_{s^{(m-1)k_0+n+3}}(x,z)+$$
$$+\frac{1}{z^{(m-2)k_0+n+3}}\sum_{P\in S_{(m-2)k_0+n+3}(2,(m-2)k_0+n+1)\setminus(1...)} b^P_{s^{(m-2)k_0+n+4}}(x,z)+...+$$
$$+\frac{1}{z^{k_0+n+m}}\sum_{P\in S_{k_0+n+m}(m-1,k_0+n+1)\setminus(1...)} b^P_{s^{k_0+n+m+1}}(x,z)+$$
$$+\frac{1}{z^{n+m+1}}\sum_{P\in S_{n+m+1}(m,n+1)} b^P_{s^{n+m+2}}(x,z)+\frac{1}{z^{(m-1)k_0+n+2}}b^{(1,2...2)}_{s^{(m-1)k_0+n+3}}(x,z)+$$
$$+\frac{1}{z^{(m-2)k_0+n+3}}\sum_{P\in S_{(m-2)k_0+n+3}(2,(m-2)k_0+n+1)\setminus(2...)} b^P_{s^{(m-2)k_0+n+4}}(x,z)+$$
$$+\frac{1}{z^{(m-3)k_0+n+4}}\sum_{P\in S_{(m-3)k_0+n+4}(3,(m-3)k_0+n+1)\setminus(2...)} b^P_{s^{(m-3)k_0+n+5}}(x,z)+...+$$
$$+\frac{1}{z^{k_0+n+m}}\sum_{P\in S_{k_0+n+m}(m-1,k_0+n+1)\setminus(2...)} b^P_{s^{k_0+n+m+1}}(x,z)+\frac{1}{z^{n+m+1}}\sum_{P\in S_{n+m+1}(m,n+1)\setminus(2...)} b^P_{s^{n+m+2}}(x,z)$$

From here we get

$$y_{mk_0+n+1}(x,z)=\frac{1}{z^{mk_0+n+1}}b^{(2)}_{s^{mk_0+n}}(x,z)+$$
$$+\frac{1}{z^{(m-1)k_0+n+2}}\sum_{P\in S_{(m-1)k_0+n+2}(1,(m-1)k_0+n+1)} b^P_{s^{(m-1)k_0+n+3}}(x,z)+$$
$$+\frac{1}{z^{(m-2)k_0+n+3}}\sum_{P\in S_{(m-2)k_0+n+3}(2,(m-2)k_0+n+1)} b^P_{s^{(m-2)k_0+n+4}}(x,z)+...+$$
$$+\frac{1}{z^{k_0+n+m}}\sum_{P\in S_{k_0+n+m}(m-1,k_0+n+1)} b^P_{s^{k_0+n+m+1}}(x,z)+$$
$$+\frac{1}{z^{n+m+1}}\sum_{P\in S_{n+m+1}(m,n+1)} b^P_{s^{n+m+2}}(x,z),\ x\geq(mk_0+n+1)\tau_2;$$
$$y_{mk_0+n+1}(x,z)=0,\ x<(mk_0+n+1)\tau_2$$

We have shown that (7) is true for a natural number $n+1$.
7. Let us prove the formula (8). Suppose that (8) is valid for all integers $\leq n$, $n\leq l_0-1$.
Based on (7), the inductive assumption of (8) and recurrent formula (6) we obtain.

$$y_{(n+1)k_0}(x,z)=\frac{1}{z^{nk_0+k_0}}b^{(2)}_{s^{nk_0+k_0+1}}(x,z)+$$
$$+\frac{1}{z^{(n-1)k_0+k_0+1}}\sum_{P\in S_{(n-1)k_0+k_0+1}(1,(n-1)k_0+k_0)} b^P_{s^{(n-1)k_0+k_0+2}}(x,z)+$$
$$+\frac{1}{z^{(n-2)k_0+k_0+2}}\sum_{P\in S_{(n-2)k_0+k_0+2}(2,(n-2)k_0+k_0)} b^P_{s^{(n-2)k_0+k_0+3}}(x,z)+$$
$$+\frac{1}{z^{(n-3)k_0+k_0+3}}\sum_{P\in S_{(n-2)k_0+k_0+3}(3,(n-2)k_0+k_0)} b^P_{s^{(n-2)k_0+4}}(x,z)+...$$

$$+\frac{1}{z^{2k_0+n-1}}\sum_{P\in S_{2k_0+n-1}(n-1,2k_0)} b^P_{s^{2k_0+n}}(x,z)+$$

$$+\frac{1}{z^{k_0+n}}\sum_{P\in S_{k_0+n}(n,k_0)} b^P_{s^{k_0+n+1}}(x,z)+\frac{1}{z^{n+1}}b^{(1)}_{s^{n+2}}(x,z)$$

8. Now it easily from (7), (8) and recurrent formula (6) mathematical induction to prove the exact formula (9). Mathematical induction has been shown to correct the following formula

$$y_{nk_0+2}(x,z)=\frac{1}{z^{nk_0+2}}b^{(2)}_{s^{nk_0+3}}(x,z)+\frac{1}{z^{(n-1)k_0+3}}\sum_{P\in S_{(n-1)k_0+3}(1,(n-1)k_0+2)} b^P_{s^{(n-1)k_0+4}}(x,z)+$$

$$\frac{1}{z^{(n-2)k_0+4}}\sum_{P\in S_{(n-2)k_0+4}(2,(m-2)k_0+2)} b^P_{s^{(n-2)k_0+5}}(x,z)+\ldots+\frac{1}{z^{k_0+n+1}}\sum_{P\in S_{k_0+n+1}(n-1,k_0+2)} b^P_{s^{k_0+n+2}}(x,z),\ x\geq(nk_0+2)\tau_2;$$

$$y_{nk_0+2}(x,z)=0,\ x<(nk_0+2)\tau_2,\ n=2,3,\ldots,l_0-1$$

9. Based on formulas (7) - (9), we get the solution of the equation (1) with the initial condition (2) to the segment $[0,\pi]$

$$y(x,z)=\sin zx+\sum_{k=1}^{l_0k_0} y_k(x,z)$$

i.e.

$$y(x,z)=\sin zx+\sum_{k=1}^{l_0k_0}\frac{1}{z^k}b^{(2)}_{s^{k+1}}(x,z)+\sum_{k=1}^{l_0}\frac{1}{z^k}b^{(1)}_{s^{k+1}}(x,z)+\sum_{k=1}^{(l_0-2)k_0}\frac{1}{z^{k+2}}\sum_{P\in S_{k+2}(2,k)} b^P_{s^{k+3}}(x,z)+\sum_{k=1}^{(l_0-3)k_0}\frac{1}{z^{k+3}}\sum_{P\in S_{k+3}(3,k)} b^P_{s^{k+4}}(x,z)+$$

$$\ldots+\sum_{k=1}^{2k_0}\frac{1}{z^{k+l_0-2}}\sum_{P\in S_{k+l_0-2}(l_0-2,k)} b^P_{s^{k+l_0-1}}(x,z)+\sum_{k=1}^{k_0}\frac{1}{z^{k+l_0-1}}\sum_{P\in S_{k+l_0-1}(l_0-1,k)} b^P_{s^{k+l_0}}(x,z)$$

Therefore, it is worth

$$y(x,z)=\sin zx+\sum_{k=1}^{l_0k_0}\frac{1}{z^k}b^{(2)}_{s^{k+1}}(x,z)+\sum_{k=1}^{l_0}\frac{1}{z^k}b^{(1)}_{s^{k+1}}(x,z)+\sum_{i=1}^{l_0-1}\sum_{j=1}^{(l_0-i)k_0}\frac{1}{z^{j+i}}\sum_{P\in S_{j+i}(i,j)} b^P_{s^{j+i+1}}(x,z)$$

which proves the theorem. □

3. Characteristic function. Asymptotic properties of the zeros characteristic function

For $x=\pi$ from (4) we get the characteristic function

$$F(z)=\sin z\pi+\sum_{k=1}^{l_0k_0}\frac{1}{z^k}b^{(2)}_{s^{k+1}}(z)+\sum_{k=1}^{l_0}\frac{1}{z^k}b^{(1)}_{s^{k+1}}(z)+\sum_{i=1}^{l_0-1}\sum_{j=1}^{(l_0-i)k_0}\frac{1}{z^{j+i}}\sum_{P\in S_{j+i}(i,j)} b^P_{s^{j+i+1}}(z) \tag{10}$$

where the

$$b^{(i)}_{s^{k+1}}(z)=b^{(i)}_{s^{k+1}}(\pi,z),\ i=1,2;\ b^P_{s^{j+i+1}}(z)=b^P_{s^{j+i+1}}(\pi,z)$$

We use the elementary identity

$$\sin z(\pi-t_1)\sin z(t_1-\tau_i-t_2)\sin z(t_2-\tau_j)=\frac{1}{4}\left[\sin z(\pi-2t_2+\tau_j-\tau_i)+\sin z(\pi-2t_1+2t_2+\tau_i-\tau_j)\right]$$

$$-\frac{1}{4}\left[\sin z(\pi-\tau_i-\tau_j)+\sin z(\pi-2t_1+\tau_j+\tau_i)\right].$$

Define the following functions

$$\beta_1^{(i,j)}(z)=\int_{\tau_1+\tau_2}^{\pi}[q_i(t_1)\int_{\tau_j}^{t_1-\tau_i}q_j(t_2)dt_2]\sin z(\pi-2t_1+\tau_j+\tau_i)dt_1$$

$$\beta_2^{(i,j)}(z)=\int_{\tau_1+\tau_2}^{\pi}[q_i(t_1)\int_{\tau_j}^{t_1-\tau_i}q_j(t_2)\sin z(\pi-2t_2+\tau_j-\tau_i)dt_2]dt_1$$

$$\beta_3^{(i,j)}(z)=\int_{\tau_1+\tau_2}^{\pi}[q_i(t_1)\int_{\tau_j}^{t_1-\tau_i}q_j(t_2)\sin z(\pi-2t_1+2t_2-\tau_j+\tau_i)dt_2]dt_1,\ i\neq j \tag{11}$$

$$\beta_k(z)=\beta_k^{(i,i)}(z),\ i=1,2,\ k=1,2,3.$$

$$a^{(i)}(z)=\int_{\tau_i}^{\pi}q_i(t_1)\cos z(\pi-2t_1+\tau_i)dt_1$$

We introduce the following sizes

$$J_1^{(i)}=\int_{\tau_i}^{\pi}q_i(t_1)dt_1,\ J_2^{(i)}=\int_{2\tau_i}^{\pi}q_i(t_1)\int_{\tau_i}^{t_1-\tau_i}q_i(t_2)dt_2dt_1,\ i=1,2 \tag{12}$$

$$J_2^{(i,j)}=\int_{\tau_1+\tau_2}^{\pi}q_i(t_1)\int_{\tau_j}^{t_1-\tau_i}q_j(t_2)dt_2dt_1,\ i\neq j$$

Function (10) with (11) and (12) can be rewritten in the form:

$$\begin{aligned}F(z)=&\sin\pi z+\frac{1}{2z}\sum_{i=1}^{2}[a^{(i)}(z)-J_1^{(i)}\cos z(\pi-\tau_i)]+\\&+\frac{1}{4z^2}\left\{\sum_{i=1}^{2}\left[\sum_{p=2}^{3}\beta_p(z)-\beta_1(z)-J_2^{(i)}\sin z(\pi-2\tau_i)\right]+\sum_{p=2}^{3}[\beta_p^{(1,2)}(z)+\beta_p^{(2,1)}(z)]\right\}\\&-\frac{1}{4z^2}\left[\beta_1^{(1,2)}(z)+\beta_1^{(2,1)}(z)+(J_2^{(1,2)}+J_2^{(2,1)})\sin z(\pi-\tau_1-\tau_2)\right]+O\left(\frac{1}{z^4}\right)\end{aligned} \tag{13}$$

Theorem 3.1. *If $q_j(x)\in L_1[0,\pi]$, $j=1,2$ then zeros z_n, $n\in N$ of the function (13) have an asymptotics shape*

$$z_n=n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right),(n\to\infty)$$

where

$$C_1(n)=\sum_{i=1}^{2}\left(\frac{J_1^{(i)}}{2\pi}\cos n\tau_i-\frac{1}{2\pi}a_n^{(i)}\right)$$

$$C_2(n)=\frac{1}{2\pi^2}\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{2}\hat{b}_n^{(i)}J_1^{(j)}\cos n\tau_j-\hat{b}_n^{(i)}a_n^{(j)}+\frac{\pi-\tau_i}{2}J_1^{(i)}J_1^{(j)}sinn\tau_i\cos n\tau_j\right)\right]+$$

$$+\frac{1}{2\pi^2}\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{2}\frac{\pi+\tau_i}{2}b_n^{(i)}a_n^{(j)}-\frac{\pi-\tau_i}{2}J_1^{(i)}a_n^{(j)}\sin n\tau_i-\frac{\pi+\tau_i}{2}J_1^{(j)}b_n^{(i)}\cos n\tau_j\right)\right]+ \qquad (**)$$

$$+\frac{1}{4\pi}\left[\sum_{i=1}^{2}\left(\sum_{p=2}^{3}b_{p,n}^{(i)}-b_{1,n}^{(i)}-J_2^{(i)}\sin 2n\tau_i\right)+\sum_{p=2}^{3}(b_{p,n}^{(1,2)}+b_{p,n}^{(2,1)})-b_{1,n}^{(1,2)}-b_{1,n}^{(2,1)}\right]-$$

$$-\frac{J_2^{(1,2)}+J_2^{(1,2)}}{4\pi}\sin n(\tau_1+\tau_2)+O\left(\sum_{i=1}^{2}\sum_{j=1}^{2}a_n^{(j)}b_n^{(i)}\right)$$

Proof. Since the $q_j(x)\in L_1[0,\pi],\ j=1,2$, it is true that $\beta_k^{(i,j)}(z)=O\left(\frac{1}{n^s}\right)\ s>\frac{1}{2}$
Take

$$z_n=n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right),(n\to\infty) \qquad (14)$$

Putting exspression (14) into the equation (13). If we include

$$\sin\pi z_n=(-1)^n\left[\frac{\pi C_1(n)}{n}+\frac{\pi C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right)\right],\ \ \frac{1}{z_n}=\frac{1}{n}+O\left(\frac{C_2(n)}{n^3}\right)$$

$$\cos z_n(\pi-\tau_i)=(-1)^n\left[\cos n\tau_i+\frac{(\pi-\tau_i)C_1(n)}{n}\sin n\tau_i+O\left(\frac{C_2(n)}{n^2}\right)\right]$$

$$\sin z_n(\pi-2\tau_i)=(-1)^{n+1}\sin 2n\tau_i+O\left(\frac{C_1(n)}{n}\right),\ \sin z_n(\pi-\tau_1-\tau_2)=(-1)^{n+1}\sin n(\tau_1+\tau_2)+O\left(\frac{C_1(n)}{n}\right)$$

Let us define the folowing number sequences:

$$a_n^{(i)}=\int_{\tau_i}^{\pi}q_i(t_1)\cos n(2t_1-\tau_i)dt_1 \quad i=1,2,\ n=1,2,...$$

$$b_n^{(i)}=\int_{\tau_i}^{\pi}q_i(t_1)\sin n(2t_1-\tau_i)dt_1,\ \hat{b}_n^{(i)}=\int_{\tau_i}^{\pi}t_1q_i(t_1)\sin n(2t_1-\tau_i)dt_1$$

Next, apply

$$a^{(i)}(z_n)=\int_{\tau_i}^{\pi}q_i(t_1)\cos\left(n+\frac{C_1(n)}{n}+\frac{C_2(n)}{n^2}+o\left(\frac{C_2(n)}{n^2}\right)\right)(\pi-2t_1+\tau_i)dt_1=$$

$$=(-1)^n\left\{a_n^{(i)}+\left[\frac{C_1(n)(\pi+\tau_i)}{n}+\frac{C_2(n)(\pi+\tau_i)}{n^2}\right]b_n^{(i)}\right\}+$$

$$+(-1)^{n+1}\left[\frac{2C_1(n)}{n}+\frac{2C_2(n)}{n^2}\right]\hat{b}_n^{(i)}+o\left(\frac{b_n^{(i)}C_2(n)}{n^2}\right)$$

Let us introduce the following sequences

$$b_{1,n}^{(i,j)}=\int_{\tau_1+\tau_2}^{\pi}[q_i(t_1)\int_{\tau_j}^{t_1-\tau_i}q_j(t_2)dt_2]\sin n(2t_1-\tau_j-\tau_i)dt_1$$

$$b_{2,n}^{(i,j)} = \int\limits_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int\limits_{\tau_j}^{t_1-\tau_i} q_j(t_2) \sin n(2t_2 - \tau_j + \tau_i) dt_2] dt_1$$

$$b_{3,n}^{(i,j)} = \int\limits_{\tau_1+\tau_2}^{\pi} [q_i(t_1) \int\limits_{\tau_j}^{t_1-\tau_i} q_j(t_2) \sin n(2t_1 - 2t_2 + \tau_j - \tau_i) dt_2] dt_1, \; n \in N$$

Then apply the relation

$$\beta_p^{(i,j)}(z_n) = (-1)^{n+1} b_{p,n}^{(i,j)} + O\left(\frac{C_1(n) b_{1,n}^{(i,j)}}{n}\right), \; p = 1, 2, 3$$

Using the previous equality, relation (13) becomes:

$$0 = F(z_n) = \frac{(-1)^n}{n}\left[\pi C_1(n) + \frac{1}{2}\sum_{i=1}^{2}(a_n^{(i)} - J_1^{(i)} \cos n\tau_i)\right] +$$

$$+\frac{(-1)^n}{n^2}\left\{\pi C_2(n) + \sum_{i=1}^{2}\left[\frac{C_1(n)(\pi+\tau_i)}{2} b_n^{(i)} - C_1(n)\hat{b}_n^{(i)} - \frac{(\pi-\tau_i)C_1(n)}{2} J_1^{(i)} \sin n\tau_i\right]\right\} -$$

$$-\frac{(-1)^n}{4n^2}\left\{\sum_{i=1}^{2}\left(\sum_{p=2}^{3} b_{p,n}^{(i)} - b_{1,n}^{(i)} - J_2^{(i)} sin 2n\tau_i\right) + \sum_{p=2}^{3}(b_{p,n}^{(1,2)} + b_{p,n}^{(2,1)})\right\} -$$

$$+\frac{(-1)^n}{4n^2}\left[b_{1,n}^{(1,2)} + b_{1,n}^{(2,1)} + (J_2^{(1,2)} + J_2^{(2,1)}) \sin n(\tau_1+\tau_2)\right] + O\left(\frac{b_n^{(i)} C_2(n)}{n^3}\right)$$

and grouping expression by degrees, we get:

$$C_1(n) = \sum_{i=1}^{2}\left(\frac{J_1^{(i)}}{2\pi} \cos n\tau_i - \frac{1}{2\pi} a_n^{(i)}\right)$$

$$C_2(n) = \frac{1}{2\pi^2}\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{2} \hat{b}_n^{(i)} J_1^{(j)} \cos n\tau_j - \hat{b}_n^{(i)} a_n^{(j)} + \frac{\pi-\tau_i}{2} J_1^{(i)} J_1^{(j)} sin n\tau_i \cos n\tau_j\right)\right] +$$

$$+\frac{1}{2\pi^2}\left[\sum_{i=1}^{2}\left(\sum_{j=1}^{2} \frac{\pi+\tau_i}{2} b_n^{(i)} a_n^{(j)} - \frac{\pi-\tau_i}{2} J_1^{(i)} a_n^{(j)} \sin n\tau_i - \frac{\pi+\tau_i}{2} J_1^{(j)} b_n^{(i)} \cos n\tau_j\right)\right] +$$

$$+\frac{1}{4\pi}\left[\sum_{i=1}^{2}\left(\sum_{p=2}^{3} b_{p,n}^{(i)} - b_{1,n}^{(i)} - J_2^{(i)} \sin 2n\tau_i\right) + \sum_{p=2}^{3}(b_{p,n}^{(1,2)} + b_{p,n}^{(2,1)}) - b_{1,n}^{(1,2)} - b_{1,n}^{(2,1)}\right] -$$

$$-\frac{J_2^{(1,2)} + J_2^{(1,2)}}{4\pi} \sin n(\tau_1+\tau_2) + O\left(\sum_{i=1}^{2}\sum_{j=1}^{2} a_n^{(j)} b_n^{(i)}\right)$$

This proves the theorem 3.2. □

Remark 3.2. *As potential $q_j(x) \in L_1[0,\pi]$, $j = 1,2$, asymptotics of zeros z_n is characterized by the complexity of the coefficients $C_1(n)$ and $C_2(n)$. In expression $C_1(n)$ summand $\frac{J_1^{(i)}}{2\pi}\cos n\tau_i$ oscillates as $n \longrightarrow \infty$, and $\frac{1}{2\pi}a_n^{(i)}$ vanishes. Similarly to the in $C_2(n)$. Expression*

$$\zeta_2 = \frac{1}{2\pi^2}\sum_{i=1}^{2}\left(\sum_{j=1}^{2}\frac{\pi-\tau_i}{2}J_1^{(i)}J_1^{(j)}\sin n\tau_i\cos n\tau_j\right) - \frac{1}{4\pi}\left[\sum_{i=1}^{2}J_2^{(i)}\sin 2n\tau_i + (J_2^{(1,2)} + J_2^{(2,1)})\sin n(\tau_1+\tau_2)\right]$$

oscillates, while the expression

$$\zeta_2^* = \frac{1}{2\pi^2}\sum_{i=1}^{2}\left[\sum_{j=1}^{2}\left(\hat{b}_n^{(i)}J_1^{(j)}\cos n\tau_j + \frac{\pi+\tau_i}{2}b_n^{(i)}a_n^{(j)} - \frac{\pi-\tau_i}{2}J_1^{(i)}a_n^{(j)}\sin n\tau_i - \hat{b}_n^{(j)}a_n^{(j)} - \frac{\pi+\tau_i}{2}J_1^{(j)}b_n^{(i)}\cos n\tau_j\right)\right]$$
$$+\frac{1}{4\pi}\left[\sum_{i=1}^{2}\left(\sum_{p=2}^{3}b_{p,n}^{(i)} - b_{1,n}^{(i)} - J_2^{(i)}\sin 2n\tau_i\right) + \sum_{p=2}^{3}(b_{p,n}^{(1,2)} + b_{p,n}^{(2,1)}) - b_{1,n}^{(1,2)} - b_{1,n}^{(2,1)}\right] + O\left(\sum_{i=1}^{2}\sum_{j=1}^{2}a_n^{(j)}b_n^{(i)}\right)$$

vanishes as $n \to \infty$.

4. Determining the regularized trace

From $\lambda_n = z_n^2$ follows $\lambda_n = n^2 + 2C_1(n) + \frac{2C_2(n)}{n} + o\left(\frac{C_2(n)}{n}\right)$

Definition 4.1. *The sum of this series*

$$s_1 = \sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{1}{\pi}\sum_{j=1}^{2}(J_1^{(j)}cosn\tau_j - a_n^{(j)})\right)$$

is called the regularized trace of first order operator D^2.

Remark 4.2. *Since $\lambda_n - n^2 - \frac{1}{\pi}\sum_{j=1}^{2}(J_1^{(j)}cosn\tau_j - a_n^{(j)}) = O\left(\frac{C_2(n)}{n}\right)$ where $C_2(n)$ is given with (**). The series $\sum_{n=1}^{\infty}\left(\lambda_n - n^2 - \frac{1}{\pi}\sum_{j=1}^{2}(J_1^{(j)}cosn\tau_j - a_n^{(j)})\right)$ converges, so the trace s_1 well defined.*

Theorem 4.3. *If $q_j(x) \in L_1[0,\pi]$, $j = 1,2$ first regularized trace operator D^2 defined by (1) is $s_1 = 0$*

Proof. The entire function (10) can be represented by its zeros z_n, $n \in N_0$ in the form of an infinite product.

$$F(z) = Az\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n}\right),\ \sin\pi z = \pi z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n}\right)$$

$$\frac{F(z)}{\sin\pi z} = \frac{A}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}\prod_{n=1}^{\infty}\left(\frac{\lambda_n - z^2}{n^2 - z^2}\right) = \frac{A}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}\prod_{n=1}^{\infty}\left(1 + \frac{\lambda_n - n^2}{n^2 - z^2}\right)$$

where A is indefinite parameter. If we put $B = \frac{A}{\pi}\prod_{n=1}^{\infty}\frac{n^2}{\lambda_n}$ then we have

$$\frac{F(z)}{\sin\pi z} = B\left[1 + \sum_{n=1}^{\infty}\frac{\lambda_n - n^2}{n^2 - z^2} + \Delta\psi(z)\right],\ z \in C\setminus Z \tag{15}$$

From (13) and (15) it follows that B=1, i.e. $A = \pi \prod_{n=1}^{\infty} \frac{\lambda_n}{n^2}$

By analogy with the Levitan transformation (see[2]), we write

$$\sum_{n=1}^{\infty} \frac{\lambda_n - n^2}{n^2 - z^2} = \sum_{n=1}^{\infty} \frac{\lambda_n - n^2 - \frac{1}{\pi}\sum_{j=1}^{2}\left(J_1^{(j)}\cos n\tau_j - a_n^{(j)}\right)}{n^2 - z^2} + \sum_{j=1}^{2}\left[\frac{J_1^{(j)}}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\tau_j}{n^2 - z^2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{a_n^{(j)}}{n^2 - z^2}\right]$$

$$= -\frac{1}{z^2}s_1 + \frac{1}{z^2}\sum_{n=1}^{\infty}\frac{n^2\left(\lambda_n - n^2 - \frac{1}{\pi}\sum_{j=1}^{2}\left(J_1^{(j)}\cos n\tau_j - a_n^{(j)}\right)\right)}{n^2 - z^2} + \sum_{j=1}^{2}\frac{J_1^{(j)}}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\tau_j}{n^2 - z^2} - \frac{1}{\pi}\sum_{j=1}^{2}\sum_{n=1}^{\infty}\frac{a_n^{(j)}}{n^2 - z^2}$$

Therefore, it is

$$\frac{1}{z^2}\sum_{n=1}^{\infty}\frac{n^2\left(\lambda_n - n^2 - \frac{1}{\pi}\sum_{j=1}^{2}\left(J_1^{(j)}\cos n\tau_j - a_n^{(j)}\right)\right)}{n^2 - z^2} = O\left(\frac{\sin z(\pi - \tau_2)}{z^2 \sin \pi z}\right) \tag{16}$$

Furthermore, based on the known relations

$$\sum_{n=1}^{\infty}\frac{\cos n\tau_j}{n^2 - z^2} = \frac{1}{2z^2} - \frac{\pi}{2z}\frac{\cos z(\pi - \tau_j)}{\sin \pi z} \tag{17}$$

$$\sum_{n=1}^{\infty}\frac{a_n^{(j)}}{n^2 - z^2} = \int_{\tau_j}^{\pi} q_j(t_1)\left(\frac{1}{2z^2} - \frac{\pi}{2z}\frac{\cos z(\pi - 2t_1 + \tau_j)}{\sin \pi z}\right)dt_1 \tag{18}$$

Also on the basis of (16), (17) and (18) we obtain

$$\sum_{n=1}^{\infty}\frac{\lambda_n - n^2}{n^2 - z^2} = -\frac{1}{z^2}s_1 - \frac{1}{2z\sin \pi z}\sum_{j=1}^{2}J_1^{(j)}\cos z(\pi - \tau_j) + \frac{1}{2z}\sum_{j=1}^{2}\frac{a^{(j)}(z)}{\sin \pi z} + O\left(\frac{\sin z(\pi - \tau_2)}{z^2 \sin \pi z}\right)$$

and finally we got expression

$$F(z) = \sin \pi z - \frac{\sin \pi z}{z^2}s_1 - \frac{1}{2z}\sum_{j=1}^{2}J_1^{(j)}\cos z(\pi - \tau_j) + \frac{1}{2z}\sum_{j=1}^{2}a^{(j)}(z) + O\left(\frac{\sin z(\pi - \tau_2)}{z^2}\right) \tag{19}$$

If we set $z = -iy$, $(y \to \infty)$ in (13) and (19) we obtain $s_1 = 0$.

The first regularized trace of the operator (1-3) is obtained analogously to the calculation of the first regularized trace in the works [5] and [7]. □

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